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Improved First-Order Approximation of Eigenvalues and Eigenvectors

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A method based on reduced basis approximation concepts is presented for improved first-order approximation of eigenvalues and eigenvectors of modified structural dynamic systems. The terms of a local approximation based on Taylor or matrix power series are used as basis vectors for approximating the perturbed eigenparameters. For each eigenmode, a reduced eigensystem is generated by using the baseline eigenvector and the first-order approximation term as Ritz vectors. The solution of the reduced eigensystem leads to two possible estimates of each perturbed eigenvalue and eigenvector. Criteria for selection of the best approximation are presented. The zero- and first-order Rayleigh quotient approximation can be directly recovered as special cases of the present method. Results are presented for approximate dynamic reanalysis of a 25-bar planar truss structure. It is shown that high-quality approximation of the perturbed eigenparameters can be obtained for moderate perturbations in the stiffness and mass matrices. For very large local perturbations of the structure, including deletion of some structural members, it is shown that the present method yields reasonable-quality approximations.

Introduction

THE problem of predicting the dynamic response characteristics of modified structures has attracted a good deal of attention in the structural dynamics and structural optimization communities. This has been primarily due to the ever-increasing demand for faster computational procedures for the optimal design of large-scale structures, including robustness analyses of structural and control systems. Furthermore, these procedures can also be applied to the task of reconciliation of analytical models with vibration test data and structural damage identification.

Traditionally, a truncated Taylor or matrix power series evaluated at a nominal design point is used to approximate the eigenparameters of modified structures (see, for example, Refs. 1 and 2). Earlier studies[3] indicated that the quality of local approximation can be improved by using the reciprocal design variables as intermediate variables in the Taylor series expansion. Woo[4] proposed a generalized hybrid constraint (GHC) approximation strategy, which uses a mix of direct and reciprocal design variables to arrive at conservative approximations. However, it is well known that the eigenparameters are highly nonlinear in both the direct and reciprocal design spaces. Hence, local-approximation concepts can be applied only for small perturbations in the stiffness and mass matrices. Even the use of higher-order terms in the local-approximation series cannot guarantee convergence for moderate to large perturbations in the structural parameters. The implication of this observation in the context of structural optimization is that severe move limits have to be imposed in line searches to ensure convergence to a feasible design.

Very few studies in the literature have addressed the structural dynamic reanalysis problem for moderate to large perturbations in the structural parameters. The approaches currently in use can be broadly classified into direct and iterative approaches. The objective of most direct approaches is to increase the range of validity of local approximation techniques. Inamura[5] proposed an approximation procedure in which the eigenpair perturbation equations are interpreted as differential equations in terms of the perturbation parameters. It was shown that excellent improvements over the local approximation can be obtained for moderate perturbations in the structural parameters. A procedure using the eigensensitivity equations was developed by Pritchard and Adelman[6] based on a similar line of approach. Canfield[7] conducted a study to determine the most fundamental intermediate variables for approximating the eigenvalues. It was shown that the use of modal strain and kinetic energy as intermediate variables results in better accuracy as compared to local approximation based on either direct or reciprocal design variables or the GHC strategy. Similar observations were made earlier by Murthy and Haftka.[8]

High[9] proposed an iterative modal method to compute the perturbations in the frequencies and mode shapes. Later studies by Eldred et al.[10] indicated that difficulties may arise in the convergence of High's method for moderate eigenpair perturbations. An improved normalization scheme was proposed to improve the convergence properties of High's method.

An iterative higher-order eigenperturbation (HOEP) procedure, which makes use of the nonlinear form of the eigenproblem perturbation equations, was developed by Eldred et al.[11] It was shown that this method converges to the exact solution for moderate to large perturbations in the system matrices of the order of 150%. An exact method based on the block Lanczos algorithm was proposed by Carey et al.[12] Even though both these procedures can provide exact results, the computational effort involved is substantial compared to the direct approaches mentioned earlier.

More recently, Balmes[13] presented a novel approach in which a finite element model is represented as a parametric family of reduced order. The full-order finite element model is reduced using a transformation matrix composed of the response vectors evaluated at different points in the design space. Extremely encouraging results were obtained for approximate static and dynamic reanalysis of a cantilevered box beam structure.

The main difference between the present method and Balmes's approach is that the basis vectors chosen here are an implicit function of the extent of perturbation in the structural parameters. In contrast, Balmes's approach[13] uses constant basis vectors that are invariant to the parametric perturbations. Also, here we seek to approximate each eigenmode independently, which is expected to result in better computational efficiency, especially for cases in which a large number of eigenpairs are to be reanalyzed or the size of the design space under consideration is large. It is also of interest to note that a procedure on similar lines was developed earlier by Kirsch[14,15] and was applied with a great deal of success to approximate static reanalysis.

This paper presents a direct solution procedure for approximate reanalysis of the eigenvalues and eigenvectors of modified

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structures. The results of a single precise analysis computed at a nominal design point are used to approximate the perturbed eigenparameters. The baseline eigenvector and the first-order approximation term are used as basis vectors along with undetermined scaling factors to approximate the perturbed eigenvector. The scaling factors are then determined via Ritz analysis of the perturbed eigenvalue problem using these basis vectors, i.e., by solving a reduced 2×2 eigensystem for each eigenmode of interest. This results in two possible estimates of the perturbed eigenpair. Criteria for selection of the best approximation are presented. The proposed method is applied to approximate dynamic reanalysis of a 25-bar planar truss structure. The results are compared with those obtained using a first-order Taylor series approximation and Rayleigh quotient approximations. It is shown that high-quality approximation of the eigenvalues and eigenvectors can be obtained for simultaneous perturbations in the stiffness and mass matrices of the order of $\pm 50\%$. For cases involving local perturbations in the stiffness and mass matrices, it is shown that the present method yields nearly exact results for very large perturbations.

Problem Statement and Conventional Methods

The free-vibration undamped natural frequencies and mode shapes of a linear structural system can be computed by solving the algebraic eigenvalue problem

$$[\boldsymbol{K}^0]\{\phi^0\}_i = \lambda_i^0[\boldsymbol{M}^0]\{\phi^0\}_i \tag{1}$$

where $[\boldsymbol{K}^0], [\boldsymbol{M}^0] \in \Re^{n \times n}$ are the structural stiffness and mass matrix, respectively. The system matrices are considered to be a general function of the design variables denoted by $\{\boldsymbol{X}\} = \{x_1, x_2, \ldots, x_p\}$, and λ_i^0 and $\{\phi^0\}_i$ are the eigenvalue and eigenvector of mode i, respectively.

Consider the case wherein the design variables are perturbed by $\{\Delta \boldsymbol{X}\}$. Let $[\Delta \boldsymbol{K}]$ and $[\Delta \boldsymbol{M}]$ be the corresponding perturbation in the stiffness and mass matrices. The perturbed eigenvalue problem can be written as

$$([\boldsymbol{K}^0] + [\Delta \boldsymbol{K}])\big(\{\phi^0\}_i + \{\Delta\phi\}_i\big) = \big(\lambda_i^0 + \Delta\lambda_i\big) \times ([\boldsymbol{M}^0] + [\Delta \boldsymbol{M}])\big(\{\phi^0\}_i + \{\Delta\phi\}_i\big) \tag{2}$$

where $\Delta\lambda_i$ and $\{\Delta\phi\}_i$ are the eigenvalue and eigenvector perturbation, respectively. Equation (2) can be written in a compact form as

$$[\boldsymbol{K}]\{\phi\}_i = \lambda_i[\boldsymbol{M}]\{\phi\}_i \tag{3}$$

Often it is found that, even for small to moderate perturbations in the stiffness and mass matrices, significant alterations in the modal characteristics of the structure may occur. Hence, an exact reanalysis becomes necessary to compute the perturbed eigenparameters with sufficient accuracy. The objective of approximate reanalysis procedures is the computation of the perturbed eigenparameters using the results of exact analysis for the baseline system without recourse to solving Eq. (3) in its exact form.

Conventional First-Order Approximation of Eigenparameters

Typically, the perturbations in the eigenparameters are calculated using first-order sensitivity information as

$$\Delta\lambda_i = \sum_{j=1}^{p} \left(\frac{\partial\lambda_i}{\partial x_j}\right)\Delta x_j \tag{4}$$

and

$$\{\Delta\phi\}_i = \sum_{j=1}^{p} \left(\frac{\partial\{\phi\}_i}{\partial x_j}\right)\Delta x_j \tag{5}$$

where $\partial\lambda_i/\partial x_j$ and $\partial\{\phi\}_i/\partial x_j$ are the sensitivities of the eigenvalues and eigenvectors with respect to the structural parameters, respectively. The eigenvalue and eigenvector derivatives can be calculated from

$$\frac{\partial\lambda_i}{\partial x_j} = \{\phi^0\}_i^T\left(\frac{\partial[\boldsymbol{K}]}{\partial x_j} - \lambda_i^0\frac{\partial[\boldsymbol{M}]}{\partial x_j}\right)\{\phi^0\}_i \tag{6}$$

$$\big([\boldsymbol{K}^0] - \lambda_i^0[\boldsymbol{M}^0]\big)\frac{\partial\{\phi\}_i}{\partial x_j} = \{\boldsymbol{F}\}_i \tag{7}$$

where

$$\{\boldsymbol{F}\}_i = \left(\lambda_i^0\frac{\partial[\boldsymbol{M}^0]}{\partial x_j} + \frac{\partial\lambda_i}{\partial x_j}[\boldsymbol{M}^0] - \frac{\partial[\boldsymbol{K}]}{\partial x_j}\right)\{\phi^0\}_i \tag{8}$$

Equations (6–8) have been derived under the assumption that the baseline eigenvectors have been mass normalized. It can seen from Eq. (6) that the computation of the eigenvalue sensitivities involves a simple and straightforward calculation. However, computation of the eigenvector sensitivities is computationally expensive and more involved because it involves the solution of the singular set of equations in Eq. (7). A solution to Eq. (7) exists because $\{\boldsymbol{F}\}_i$ is orthogonal to $\{\phi^0\}_i$. A unique solution to this equation is calculated by making use of mass normalization conditions. There exists a wealth of methods in the literature for eigensensitivity analysis (see, for example, Refs. 16–18). A comparison study of some of the methods can be found in Ref. 19. In the present study, Nelson's method[17] is employed to solve Eq. (7) and compute the eigenvector derivatives. Note that, when the derivatives of more than one eigenvector are desired, it would be computationally more efficient to make use of iterative schemes of the form proposed in Ref. 20.

Repeated Eigenvalues

When the baseline structure has repeated eigenvalues, the corresponding eigenvectors are not unique. Also because the coefficient matrix in Eq. (7) is rank deficient by a number greater than one, the methods mentioned earlier cannot be directly applied. Hence, to find the unique eigenvectors that are differentiable, an orthogonal transformation has to be employed to reorient the eigenvectors. The formulation presented henceforth will remain unchanged for cases with repeated eigenvalues. The only difference in the solution procedure would be to replace the baseline eigenvectors with the reoriented eigenvectors. Further details on eigensensitivity analysis procedures for structures with repeated eigenvalues can be found in the literature.[21,22]

Improved First-Order Approximation

The proposed approximation procedure involves the use of the baseline eigenvector and the first-order approximation term as basis vectors for Ritz analysis of the perturbed eigenvalue problem.

An assumption is made that the eigenvector of the perturbed system can be approximated in the subspace spanned by $\{\phi^0\}_i$ and $\{\Delta\phi\}_i$, which is computed using Eqs. (5–8), i.e., an approximation for the perturbed eigenvector can be written as

$$\{\hat{\phi}\}_i = \zeta_1\{\phi^0\}_i + \zeta_2\{\Delta\phi\}_i \tag{9}$$

where ζ_1 and ζ_2 are the undetermined scalar quantities in the approximate representation of the perturbed eigenvector. The assumption implicit in this proposition is that, even for moderate to large perturbations in the structural parameters, the first-order approximation yields a $\Delta\phi_i$ vector, which usually gives a reasonable indication of the likely change of the baseline eigenvector, although the magnitude or even direction of change may be erroneous. Equation (9) can be expressed in matrix form as

$$\{\hat{\phi}\}_i = [\boldsymbol{T}]\{Z\} \tag{10}$$

where $[\boldsymbol{T}] = [\phi_i^0, \Delta\phi_i] \in \Re^{n \times 2}$ and $\{Z\}^T = \{\zeta_1, \zeta_2\} \in \Re^{1 \times 2}$.

Substituting Eq. (10) into Eq. (3) and premultiplying by $[\boldsymbol{T}]^T$, the resulting set of equations can be expressed as

$$[\boldsymbol{K}_T]\{Z\} = \lambda[\boldsymbol{M}_T]\{Z\} \tag{11}$$

where

$$[\boldsymbol{K}_T] = [\boldsymbol{T}]^T[\boldsymbol{K}][\boldsymbol{T}] \in \Re^{2 \times 2} \tag{12}$$

and

$$[\boldsymbol{M}_T] = [\boldsymbol{T}]^T[\boldsymbol{M}][\boldsymbol{T}] \in \Re^{2 \times 2} \tag{13}$$

Hence, the original $n \times n$ eigensystem is represented by a reduced 2×2 eigensystem for each eigenmode to be approximated.

A nontrivial solution to the scaling parameters, i.e., $\{Z\}$, can be obtained only when λ is an eigenvalue of the matrix pair ($[\boldsymbol{K}_T]$, $[\boldsymbol{M}_T]$). Hence, an approximation for the eigenvalue of the perturbed system can be computed by solving for the roots of the quadratic

$$a\hat{\lambda}_i^2 + b\hat{\lambda}_i + c = 0 \tag{14}$$

where

$$a = m_{11}m_{22} - m_{12}^2 \tag{15}$$

$$b = 2k_{12}m_{12} - k_{11}m_{22} - m_{11}k_{22} \tag{16}$$

$$c = k_{11}k_{22} - k_{12}^2 \tag{17}$$

and k_{ij} and m_{ij} are the elements of the reduced stiffness and mass matrices ($[\boldsymbol{K}_T]$ and $[\boldsymbol{M}_T]$), respectively, the elements of which are given as

$$k_{11} = \{\phi^0\}_i^T[\boldsymbol{K}]\{\phi^0\}_i, \qquad m_{11} = \{\phi^0\}_i^T[\boldsymbol{M}]\{\phi^0\}_i$$

$$k_{12} = \{\phi^0\}_i^T[\boldsymbol{K}]\{\Delta\phi\}_i, \qquad m_{12} = \{\phi^0\}_i^T[\boldsymbol{M}]\{\Delta\phi\}_i$$

$$k_{22} = \{\Delta\phi\}_i^T[\boldsymbol{K}]\{\Delta\phi\}_i, \qquad m_{22} = \{\Delta\phi\}_i^T[\boldsymbol{M}]\{\Delta\phi\}_i$$

Solution of this quadratic give two values for the perturbed eigenvalue, for example, $\hat{\lambda}_i^{\min}$ and $\hat{\lambda}_i^{\max}$. Because the transformed matrices $[\boldsymbol{K}_T]$ and $[\boldsymbol{M}_T]$ are real and symmetric, both $\hat{\lambda}_i^{\min}$ and $\hat{\lambda}_i^{\max}$ will be real. The coefficients ζ_1 and ζ_2 corresponding to these roots can then be obtained from

$$\zeta_1 = 1.0 \tag{18}$$

and

$$\zeta_2 = \frac{k_{11} - \hat{\lambda}_i m_{11}}{k_{12} - \hat{\lambda}_i m_{12}} \tag{19}$$

Hence, the mass normalized perturbed eigenvector can be written as

$$\{\hat{\phi}\}_i = \frac{1}{\{Z\}^T[\boldsymbol{M}_T]\{Z\}}\left[\{\phi^0\}_i - \frac{k_{11} - \hat{\lambda}_i m_{11}}{k_{12} - \hat{\lambda}_i m_{12}}\{\Delta\phi\}_i\right] \tag{20}$$

Criteria for Selection of Best Approximation

Now the question arises regarding which root to choose as the best approximation for the perturbed eigenmode of interest. From the inclusion principle, the following inequality relationship can be established:

$$\hat{\lambda}_i^{\min} \le \lambda_i^{\text{rqa0}} \le \hat{\lambda}_i^{\max} \tag{21}$$

where λ_i^{rqa0} is the zero-order Rayleigh quotient approximation (RQA0), which is defined as

$$\lambda_i^{\text{rqa0}} = \frac{\{\phi^0\}_i^T[\boldsymbol{K}]\{\phi^0\}_i}{\{\phi^0\}_i^T[\boldsymbol{M}]\{\phi^0\}_i} \tag{22}$$

Clearly, the root with the minimum magnitude should give the best approximation for the fundamental eigenmode. The root corresponding to maximum contribution from the baseline eigenvector is generally expected to give reasonable approximations for the other modes, i.e., the root with maximum value of $|\zeta_1/\zeta_2|$. However, this observation was not found to apply for the higher modes, especially when large perturbations lead to mode crossing. Hence, additional heuristics have to be introduced to ensure conservative approximations. Five heuristic criteria are proposed to ensure conservative approximations. The root that satisfies most of these criteria is selected as the best approximation for that mode. These heuristic criteria are as follows: 1) maximum value of $|\zeta_1/\zeta_2|$, 2) minimum distance from the zero-order Rayleigh quotient λ_i^{rqa0}, 3) minimum distance from λ_i^0, 4) minimum magnitude, and 5) minimum distance from the root selected for the previous mode.

Analogies with Other Approximation Procedures

The proposed approximation procedure could also be interpreted as an improved Rayleigh quotient approximation procedure with one free parameter, i.e., ζ_2/ζ_1. It is interesting to note that such procedures have been applied earlier with a great deal of success to the determination of the fundamental frequency of continuous structures.[23] In fact, if the reanalysis problem were set up as finding the stationary values of a Rayleigh quotient for each eigenmode, an identical set of equations of the form derived earlier would result.

The eigenvalue approximation for mode i using the presented approach can be rewritten as

$$\hat{\lambda}_i = \frac{\left\{\zeta_1\{\phi^0\}_i + \zeta_2\{\Delta\phi\}_i\right\}^T[\boldsymbol{K}]\left\{\zeta_1\{\phi^0\}_i + \zeta_2\{\Delta\phi\}_i\right\}}{\left\{\zeta_1\{\phi^0\}_i + \zeta_2\{\Delta\phi\}_i\right\}^T[\boldsymbol{M}]\left\{\zeta_1\{\phi^0\}_i + \zeta_2\{\Delta\phi\}_i\right\}} \tag{23}$$

It can be seen that, for $\zeta_1 = \zeta_2$, the present method yields the first-order Rayleigh quotient approximation (RQA1) for the perturbed eigenvalue and the first-order Taylor series for the perturbed eigenvector. The RQA1 is

$$\lambda_i^{\text{rqa1}} = \frac{\left\{\{\phi^0\}_i + \{\Delta\phi\}_i\right\}^T[\boldsymbol{K}]\left\{\{\phi^0\}_i + \{\Delta\phi\}_i\right\}}{\left\{\{\phi^0\}_i + \{\Delta\phi\}_i\right\}^T[\boldsymbol{M}]\left\{\{\phi^0\}_i + \{\Delta\phi\}_i\right\}} \tag{24}$$

If $\{\Delta\phi\}_i$ is considered to be a very small quantity or the value of ζ_2 is taken as zero, the present procedure reduces to the RQA0 defined earlier. The assumption made in this case is that the mode shapes are invariant to the parametric perturbations.

It is of interest to note that an improved normalization scheme proposed earlier by Eldred et al.[11] was also developed on the line of approach used here. Eldred's method involved scaling of the first-order approximation of the eigenvector perturbation. The scaling factor was determined by enforcing the normalization condition with respect to the perturbed mass matrix. This results in a quadratic that may have imaginary roots. In contrast, because the present procedure involves determination of the scaling parameters via Ritz analysis of the perturbed eigenvalue problem, real scaling factors are guaranteed. Hence, the present procedure can also be viewed as an improved eigenvector normalization scheme for iterative computations.

Even though eigenvector orthogonality conditions are not explicitly enforced in the present approach, numerical experiments on test problems indicate that the present method tends to maintain the orthogonality relationships better than Taylor series approximation.

Computational Aspects

Once the eigensolution and sensitivity analysis of the baseline structure for the eigenmodes of interest has been carried out, the steps involved in the proposed procedure can be summarized as follows:

1) The perturbations in the stiffness and mass matrices $[\Delta\boldsymbol{K}]$ and $[\Delta\boldsymbol{M}]$ are computed either using first-order sensitivity information or via assembly of the system matrices of the perturbed structure.

2) The first-order approximation of the perturbation in the eigenvector is calculated using Eqs. (5–8), and the transformation matrix $[\boldsymbol{T}]$ is formed.

3) The elements of the reduced stiffness and mass matrices $[\boldsymbol{K}_T]$ and $[\boldsymbol{M}_T]$ are calculated using Eqs. (12) and (13).

4) The coefficients of the quadratic are calculated, following which the approximate eigenvalues and eigenvectors are evaluated for each eigenmode.

5) The best approximation is then estimated using the criteria discussed earlier.

It can be observed from the formulation that an approximation for the eigenvalues and eigenvectors of the perturbed system can be calculated by solving for the roots of a quadratic for each eigenmode of interest. The coefficients of the quadratic equation can be easily calculated after the first-order approximation of the perturbed eigenvector is computed. It can readily be shown that the operation count involved in calculation of the coefficients of the quadratic is of the order of n^2 for each eigenmode (assuming a fully populated stiffness and mass matrix). This implies that the computational cost of the method is one order less than that required for decomposition of the

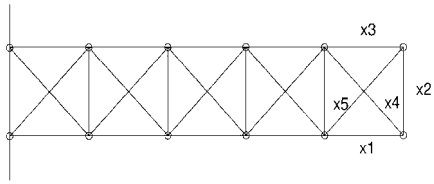

Fig. 1 Planar truss structure, five bay.

perturbed stiffness matrix. Hence, the proposed procedure involves only a few additional computations when compared to conventional first-order local approximation.

To improve the accuracy of the procedure, it may be desirable to make use of second-order approximation terms, i.e., three basis vectors, at the cost of increased computations. It is shown later that the use of two basis vectors alone can lead to good results for moderate perturbations in the stiffness and mass matrices. The proposed procedure could also be used to initialize iterative procedures, which could lead to faster convergence to the exact values of the perturbed eigenparameters.

Demonstration Examples

The proposed procedure is applied to approximate dynamic reanalysis of the 25-bar planar truss structure shown in Fig. 1. The length and height of each bay is taken as 1 m. Young's modulus and the mass density are $E = 2.1 \times 10^{11}$ N/m^2 and $\rho = 7.85 \times 10^3$ kg/m^3, respectively. A finite element model of the structure was developed using rod elements with a total of 20 degrees of freedom. A lumped mass matrix formulation was used in the analysis.

The design variables chosen are the cross-sectional areas of the truss members. For simplicity of analysis, the design variables are linked into five groups for each bay, as shown in Fig. 1. For the baseline structure, all five of the variables are taken to be equal to 0.01 m^2. Corresponding to these values of the areas, the baseline eigenparameters and their sensitivities are determined for the first four eigenmodes.

Results are presented for two cases. The first case involves simultaneous perturbations in all five of the groups of variables. In the second case, the effect of local perturbations are studied. For both these cases, the results obtained using the present approach (PA) are compared with those obtained using a first-order Taylor series (TS1), RQA1, and RQA0.

To evaluate the accuracy of the approximation, two error indices are defined. The eigenvector error (EVE) index for eigenmode i is defined as

$$\mathrm{EVE}_i = \frac{\|\{\hat{\phi}\}_i - \{\phi\}_{i\,\mathrm{exact}}\|_f}{\sqrt{\|\{\hat{\phi}\}_i\|_f \|\{\phi\}_{i\,\mathrm{exact}}\|_f}} \tag{25}$$

where $\|\{\}\|_f$ is the Frobenius norm of the vector $\{\}$. The second index is the percentage error in approximation of the ith eigenvalue.

Results and Discussion

Case 1: Simultaneous Perturbations

The effect of simultaneous perturbations in the design variables on the approximation procedure is studied first. The perturbed design variable vector is written in the form

$$\boldsymbol{X} = \boldsymbol{X}^0 + 0.01t[2.0, -0.5, 2.0, -0.5, 2.0]$$

where $\boldsymbol{X}^0 = [0.01, 0.01, 0.01, 0.01, 0.01]$.

The preceding equations imply that the perturbations in the structural parameters are studied in the range from −50 to 200% when t is varied between 0 and 1. Note that, for this case, the structure is constrained to be periodic throughout the range of perturbations as similar perturbations are applied to each bay. Figures 2–5 presents the error in approximation of the first four eigenvalues for increasing values of t. The eigenvector approximation errors for different values of t are summarized in Table 1. The following observations can be made from these results:

1) As expected, TS1, RQA1, and RQA0 provide reasonable approximations only for small values of t. For large values of t, the approximation errors for all of the methods increase. As compared to TS1 and RQA1, the approximation errors using RQA0 ultimately increase less rapidly as the value of t increases. The eigenvalue approximations using RQA1 compare fairly well with the PA for values of t up to 0.4.

2) It can seen from the results that the PA gives good-quality approximations for all four of the eigenmodes for values of t up to 0.8. This corresponds to simultaneous perturbations in all of the design variables in the range from −40 to 160%. When t increases beyond

Table 1 Eigenvector approximation error norms

	EVE_1		EVE_2		EVE_3		EVE_4	
t	PA	TS1	PA	TS1	PA	TS1	PA	TS1
0.25	0.015	0.0028	0.0269	0.0561	0.0141	0.0528	0.0220	0.0440
0.50	0.0051	0.0111	0.0789	0.1938	0.0439	0.1832	0.0791	0.1566
0.75	0.0103	0.0253	0.1381	0.3842	0.0806	0.3680	0.1720	0.3189
1.00	0.0178	0.0478	0.2017	0.6010	0.1212	0.5833	0.3010	0.5072

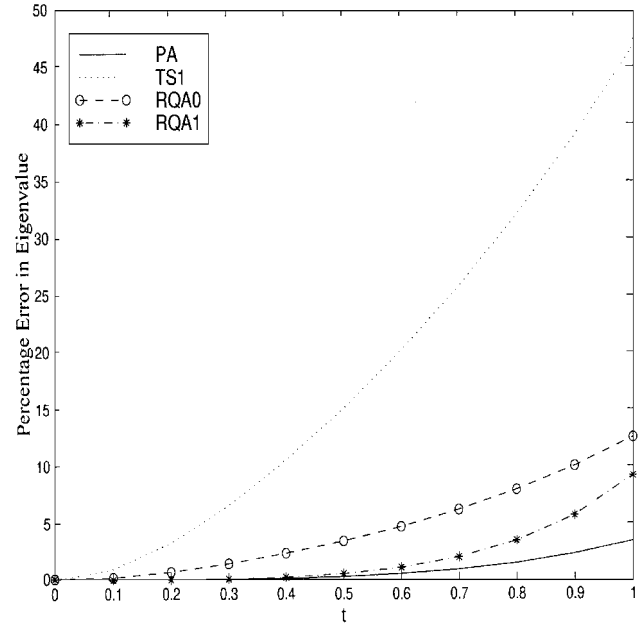

Fig. 2 Comparison of approximation errors in first eigenvalue for case 1.

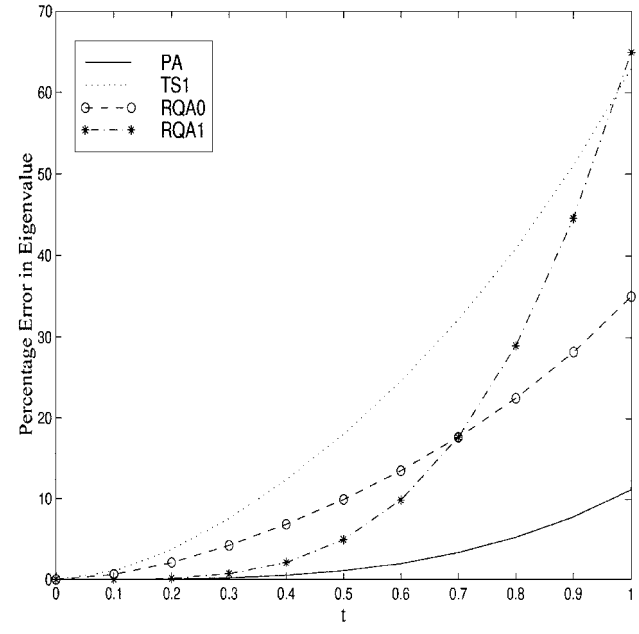

Fig. 3 Comparison of approximation errors in second eigenvalue for case 1.

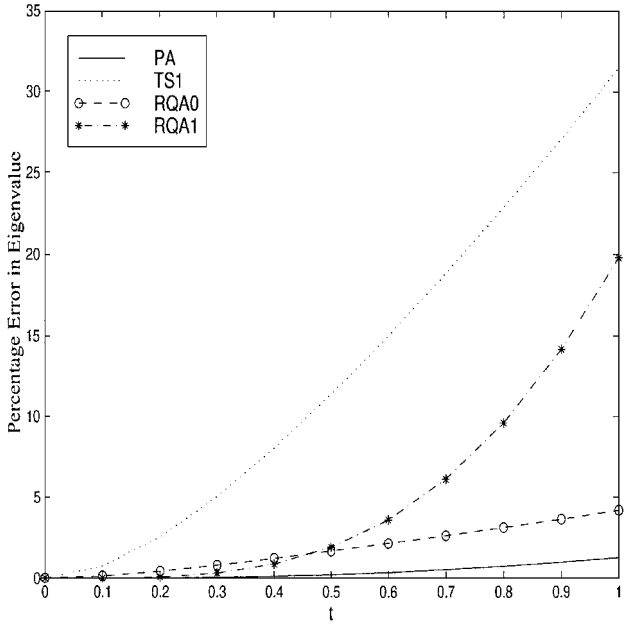

Fig. 4 Comparison of approximation errors in third eigenvalue for case 1.

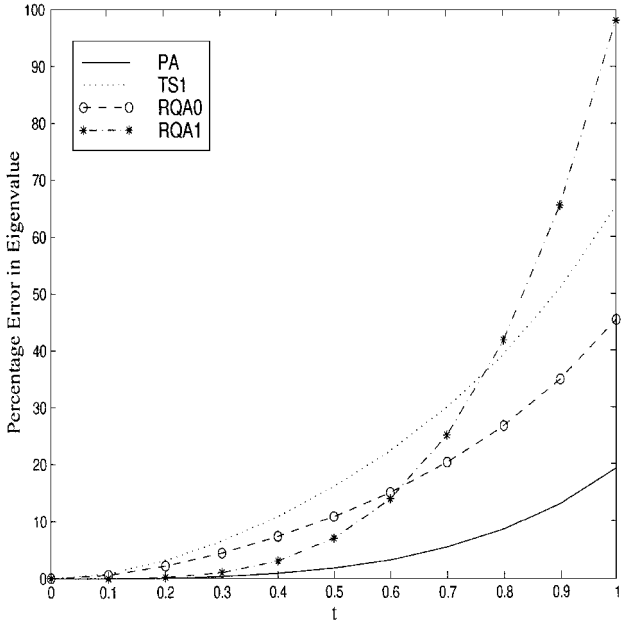

Fig. 5 Comparison of approximation errors in fourth eigenvalue for case 1.

this, errors in the second and fourth eigenmode increase considerably. However, fairly accurate approximations were obtained for the first and third eigenmodes throughout the range of perturbations considered here.

3) It can also be seen from the trends in the EVE norms (Table 1) that, using the present approach, the errors in approximation of the eigenvectors have been reduced considerably. Figure 6 compares the shapes of the exact and approximated eigenvectors at $t = 1.0$. In general, it was found that the differences between the mode shapes obtained using the PA and TS1 are quite small. However, the PA appears to be closer to the exact eigenvector in terms of the sign changes and magnitude in the various eigenvector components as indicated by the value of the EVE.

4) Figure 7 compares the frequency-response function calculated using the eigenparameter approximations with results using the exact eigensolution at the design point $t = 0.8$. The displacement response is computed for the horizontal displacement of joint 10 when the structure is subject to transverse (vertical) excitation at joint 1.

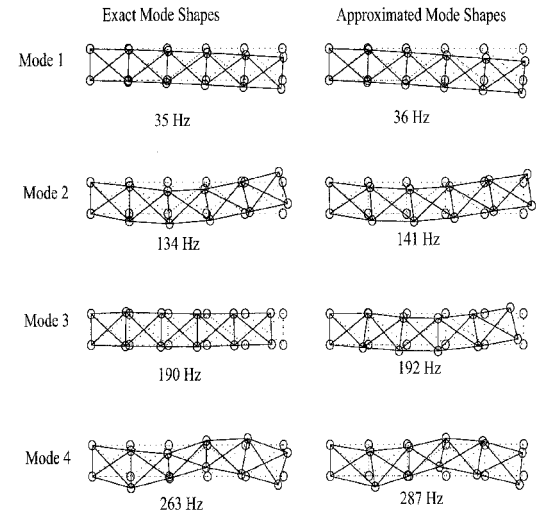

Fig. 6 Comparison of the exact and approximated mode shapes at $t = 1.0$ for case 1.

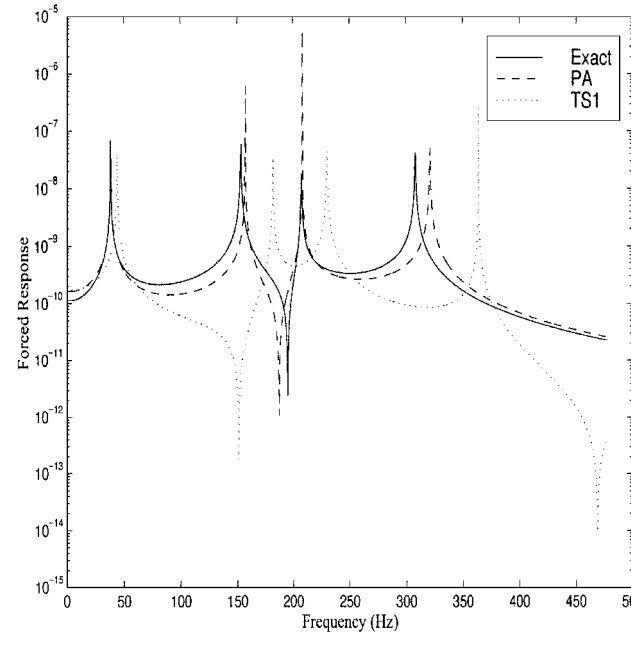

Fig. 7 Comparison of the approximated displacement response at $t = 0.8$ for case 1.

Modal damping of 0.5% has been assumed for these computations. It can be seen from Fig. 7 that a reasonable approximation of the frequency response has been achieved using the PA. In comparison, the errors found when using TS1 are substantial.

Case 2: Local Perturbations

The effect of perturbations of variable $x1$ in bay 1, $x2$ in bay 2, $x3$ in bay 3, $x4$ in bay 4, and $x5$ in bay 5 is next studied in the range from -100 to 300%. The perturbed design variable vector is written as

$$\boldsymbol{X} = \boldsymbol{X}^0 + 0.01t\left[1.0, -\tfrac{1}{3}, 1.0, -\tfrac{1}{3}, 1.0\right]$$

Note that, at $t = 3$, the structural members $x2$ and $x4$ in bays 2 and 4 are deleted from the baseline structure. A comparison of average approximation error in the first four eigenvalues for the various methods are shown in Fig. 8 as the parameter t is varied from 0 to 3. The average approximation error in the first four eigenvectors for the PA and TS1 are shown in Fig. 9. The following observations may be made:

1) As noted earlier, TS1, RQA1, and RQA0 can only approximate the perturbed eigenvalues for small changes in the system matrices.

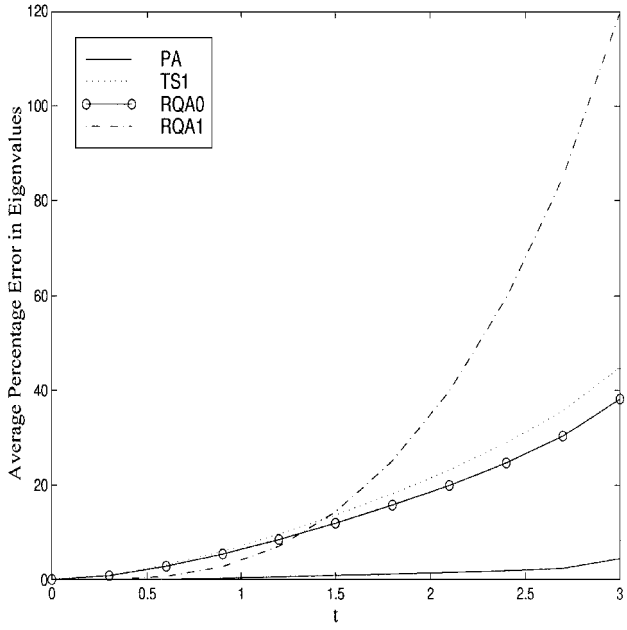

Fig. 8 Comparison of average approximation errors in first four eigenvalues for case 2.

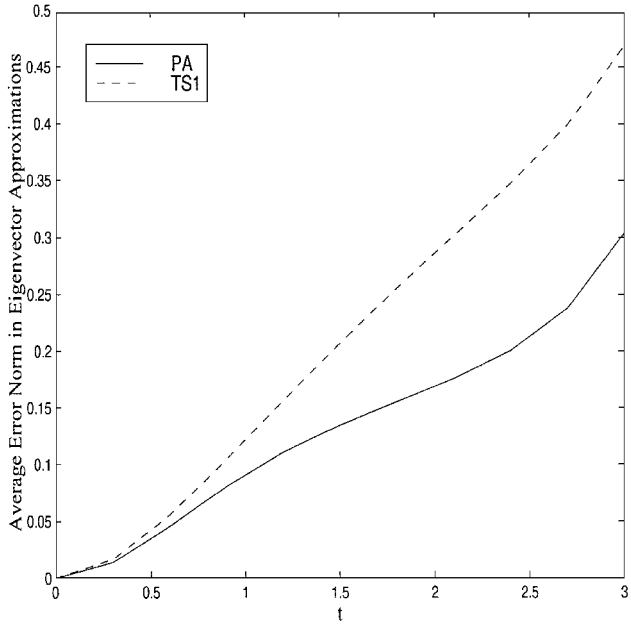

Fig. 9 Comparison of average approximation errors in first four eigenvectors for case 2.

2) It can be seen from the results that the PA gives good approximations throughout the range of perturbations considered. It is interesting to note that, even though the parameter t takes the value of 3 at one extreme, i.e., two members are deleted from the structure, the average errors in the eigenvalue approximations are of the order of just 5%. (The eigenvalue approximation errors for the first four modes were 2.3, 4.6, 2.2, and 8.7%, respectively, at $t = 3$.)

3) Figure 10 compares the exact and approximated mode shapes of the first four eigenmodes at $t = 3.0$. As mentioned earlier for case 1, no significant differences exist between the approximated mode shape using the PA and TS1. However, the PA gives considerably better approximations in terms of the EVE norm; see Fig. 9.

4) Figure 11 compares the approximated frequency response using TS1 and the PA for $t = 2.5$. The loading case considered is the same as in case 1. It can be seen that, using the PA, a significantly better approximation can again be obtained as compared to TS1.

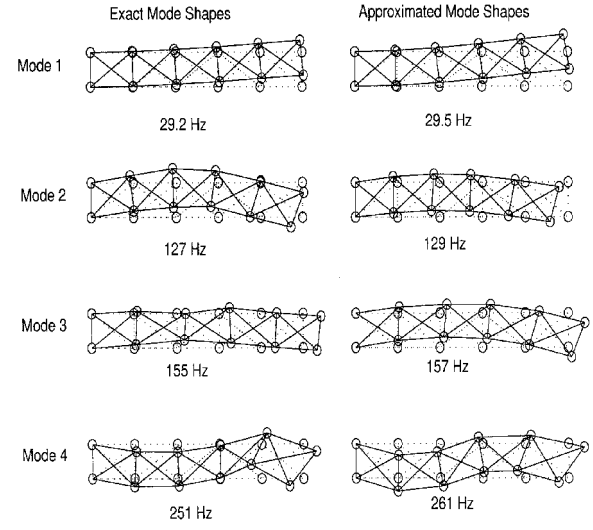

Fig. 10 Comparison of the exact and approximated mode shapes at $t = 3.0$ for case 2.

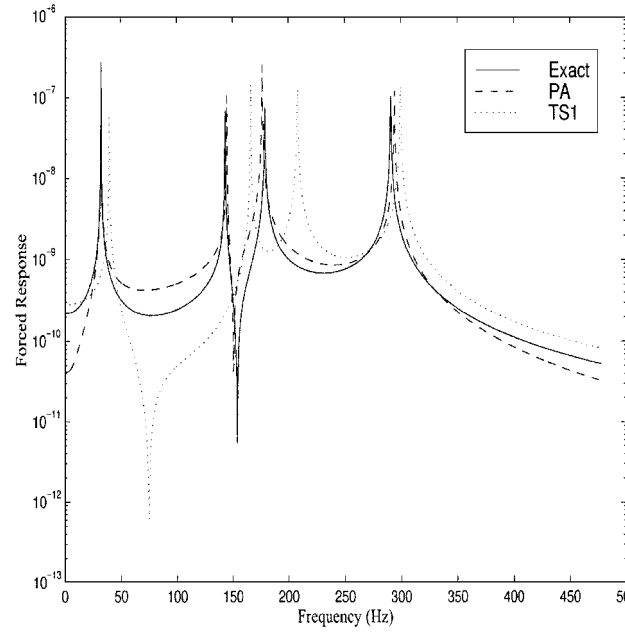

Fig. 11 Comparison of the approximated displacement response at $t = 2.5$ for case 2.

Concluding Remarks

An improved first-order approximation procedure for reanalysis of eigenvalues and eigenvectors of modified structural dynamical systems has been presented. It is shown via two demonstration examples that the PA can be used to obtain reliable estimates of the natural frequencies, mode shapes, and forced responses for simultaneous perturbation in structural parameters of the order from -40 to $+160$%. Studies on the effect of local perturbations in the structure indicate that good results can be obtained for very large changes, including deletion of some structural members. Detailed comparison studies with other local approximation techniques indicate that significant improvements are achieved with a relatively small extra computational effort. (However, further testing of the present method is required for structures with more complex modal behavior.)

It is expected that the present formulation may find applications in the areas of structural optimization and identification. Note here that extension of the PA to approximate eigensensitivity analysis is straightforward and could lead to computationally efficient procedures for structural design with dynamic response constraints. However, more basis vectors might be required for ensuring accuracy in the approximate sensitivity analysis procedure. The choice

of additional basis vectors is an open research topic. Approaches for interweaving the present method into an iterative scheme of the form presented by other investigators also merit consideration.

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